On the large time asymptotics of decaying Burgers turbulence.

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Abstract

The decay of Burgers turbulence with compactly supported Gaussian "white noise" initial conditions is studied in the limit of vanishing viscosity and large time. Probability distribution functions and moments for both velocities and velocity differences are computed exactly, together with the "time-like" structure functions $T_n(t,\tau) \equiv \langle (u(t+\tau) - u(t))^n \rangle$.

The analysis of the answers reveals both well known features of Burgers turbulence, such as the presence of dissipative anomaly, the extreme anomalous scaling of the velocity structure functions and self similarity of the statistics of the velocity field, and new features such as the extreme anomalous scaling of the "time-like" structure functions and the non-existence of a global inertial scale due to multiscaling of the Burgers velocity field.

We also observe that all the results can be recovered using the one point probability distribution function of the shock strength and discuss the implications of this fact for Burgers turbulence in general.

1 Introduction.

The study of decaying Burgers turbulence (DBT) is largely motivated by the the observation that this is a system which falls into the phenomenological class of turbulent systems which can be treated in principle by means of Kolmogorov theory. Yet the answers which can be derived analytically for Burgers turbulence are in the sharp contradiction to the predictions of Kolmogorov theory. The understanding of the reasons for such a discrepancy and their relevance for the general theory of turbulence is one of the major aims of the study of Burgers turbulence.

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The history of the subject (see e.g. [11], [23], [27], [32], [16], [28], [19], [2], [3], [4], [1], [5], [6], [18], [22], [21], [33], [34], [9], [24], [31], [35], [8]; see [17] for a review) shows however that the problem is hard, so hard in fact that it has a tendency to become self justifying, getting more and more alienated from the main body of turbulent research. However, until recently there existed no model of Burgers turbulence which can be used as a testing ground for general phenomenological theories of turbulence on one hand and admits a complete and simple analytical treatment on the other.

In present paper we introduce and analyse such a model. Namely, we study the decay of Burgers turbulence with compactly supported Gaussian "white noise" initial conditions. In physical terms the turbulence in our model is excited by an initial disturbance localized at a fixed scale much less than the size of reservoir and which can occur with equal probability around any point of the reservoir. Note that DBT driven by "white noise" plays a special role for the theory of DBT in general. The reason is that integral scale of turbulence in this problem is not imposed by initial conditions but rather is generated by time evolution. Thus, the answers one obtains for "white noise" DBT are in some sense universal. Consider for example DBT driven by Gaussian initial conditions characterized by the two point function $\chi(r)$ which is approximately constant for $r << R$ and goes to 0 exponentially fast for $r >> R$. Then the statistics of the velocity field in this model at scales much larger than R and much less than the integral scale is asymptotically equivalent, in the limit as $\nu \to 0, t \to \infty$, to that of "white noise" DBT. Likewise, compactly supported "white noise" DBT defines a universality class of models of DBT driven by compactly supported Gaussian initial conditions.

The choice of a simple initial condition and the choice to look for answers only in the vanishing viscosity and large time limits lead to a model that is exactly solvable. Explicit asymptotics can be obtained for statistics that are hard to estimate in more general models. The main reason for the exact solvability of our model is the fact that the statistics of the velocity field in the case of compactly supported initial conditions are dominated in the limit $\nu \to 0, t \to \infty$ by two shock configurations, the statistics of which is easily computable as functionals of white noise.

We would like to stress that our model is in a different universality class than the original Burgers model in which turbulence is initiated by white noise initial conditions but no restriction of compactness is imposed: a solution to Burgers equation corresponding to an initial condition supported on a whole line will generically contain infinitely many shocks at any moment of time, not just two as in our case. Accordingly, the large time statistics of the velocity field in our case is very different from that in Burgers' model. For instance, energy density decays as $t^{-1/2}$ in our case (see section 3.1) and as $t^{-2/3}$ in Burgers', [11].

The paper is organized as follows. In section 2 we give a precise statement of the problem, construct a large time limit of the solution to the inviscid Burgers equation corresponding to compactly supported initial conditions and formulate the main statements about the statistics of these solutions. In section 3 we obtain asymptotics for a variety of statistics: the moments of velocity field; the probability distribution function of velocities; the velocity structure func-

tions; the probability distribution function of velocity differences; time-like velocity structure functions. In section 4 the analysis of these results is given. In particular, the validity of one shock approximation and multiscaling in the problem are discussed.

2 The limiting velocity field.

Consider the following initial value problem connected to the Burgers equation:

$$\frac{\partial u}{\partial t}(x,t)+u(x,t)\frac{\partial u}{\partial x}(x,t)=\nu\frac{\partial^2 u}{\partial x^2}(x,t),\ \ x\in\mathbf{R},\ \ t>0, \tag{1}$$

$$u(x,0)=u_0(x), \tag{2}$$

where $u_0(x)$ is a bounded function which is compactly supported in the interval $[x_0-l,x_0+l]$. Here l is a fixed positive constant and x_0 is a random variable uniformly distributed in the interval $[-L,L]$. The fixed positive constant L plays a role of normalization length. Conditional on x_0 the initial velocity $u_0(x)$ will be a white noise over the interval $[x_0-l,x_0+l]$, so that it has a formal density

$$P(u_0|x_0)=\frac{1}{Z}e^{-\frac{1}{2J}\int_{x_0-l}^{x_0+l}u_0^2(x)dx}, \tag{3}$$

where Z is a normalization constant chosen in such a way that, formally,

$$\int_{-L}^{L}\frac{dx_0}{2L}\int P(u_0|x_0)D(u_0)=1.$$

J, the Gaussian variance, is a positive constant which plays a role of Loitsansky integral for the problem at hand.

Since we have a compact initial condition the distribution of the velocities $u_t(x)$ are not translation invariant. The role of x_0 is to randomise the location of the initial disturbance uniformly over the interval $[-L,L]$. The values of $u_t(x)$ at a fixed x will then typically be non-zero only with probability $O(L^{-1})$. We take the limit as $L\to\infty$ and all the answers concerning the statistics of the velocity field will be expressed in the form of the leading term in an asymptotic expansion in L^{-1}. This has the advantage that the answers are then translation invariant and we are free to consider statistics centered at the origin.

In what follows we will compute asymptotics of the following statistics: the moments of velocity distribution $M_n=\langle u^n(x,t)\rangle$; the velocity structure functions $S_n(y)=\langle(u(x+y,t)-u(x,t))^n\rangle$, the probability distribution function of the velocity field $P(u)=\langle\theta(u-u(x,t))\rangle$; the probability distribution function of velocity differences $P(u,y)=\langle\theta(u-u(x+y,t)+u(x,t))\rangle$; and the "time-like" velocity structure functions, $T_n(\tau,t)=\langle(u(x,t+\tau)-u(x,t))^n\rangle$. Here $\theta(z)=\chi(z\geq 0)$ is the Heavyside function and $\langle\ldots\rangle$ denotes the average w.r.t. to the random initial velocity field $u_0(x)$.

The solution of the initial value problem (1), (2) for $\nu>0$ via the Cole-Hopf transformation and the evaluation of the limit as $\nu\to 0$ for fixed $t>0$ are well known. We refer the

reader to [20] and [11] for a detailed description and give here a quick summary, sufficient for our needs. The vanishing viscosity solution can be obtained by plotting a chain of parabolic arcs such that each is touching the graph of the function $-q(x) = -\int_{-\infty}^{x} u_0(y)dy$ at two points exactly. The i-th parabolic arc is given by a graph of the function $\Phi_i(x,t) = \Phi_i + \frac{(x-x_i)^2}{2t}$. As time grows the parabolic arcs flatten out and merge, and there exists a time T^* such that for any $t > T^*$ there are generically only two arcs left. The velocity field associated with such a configuration is then given by

$$u^*(x,t) = U(x_0+x^*, x, t, P, Q) \equiv \frac{(x-x_0-x^*)}{t} \chi_{[(x_0+x^*-\sqrt{-2Qt}, x_0+x^*+\sqrt{2(P-Q)t}]}(x), \qquad (4)$$

where χ_I is an indicator function of the interval I, $P = q(+\infty)$ is a momentum corresponding to a given u_0, $Q = \min_x q(x)$ is a global minimum of $q(x)$ and $x_0 + x^* \in [x_0 - l, x_0 + l]$ is the point where this minimum is achieved. (Such a point exists and is unique almost surely as $q(x)$ is continuous and the global minimum is almost surely unique.) The limiting solution (4) was originally constructed in [20].

The time T^* at which the limiting velocity field u^* is attained depends on the random initial condition u_0 but it will be shown that the statistics of the velocity field is well approximated at large times by the statistics of the limiting velocity field u^*. The latter is determined in turn by the joint distribution of the momentum P and the global minimum Q. Indeed although the expression for u^* depends explicitly on (P, Q, x^*), the dependence on x^* doesn't influence the statistics of u^* in the limit $L \to \infty$, where the translational invariance is restored. We delegate the detailed discussion of this point to the next section.

The choice of white noise as an initial distribution leads to the distribution of the pair (P, Q) being exactly calculable. Indeed it is a well known consequence of the 'reflection principle' for Brownian paths [29]. Since it is key to all our asymptotics we include a quick derivation of the joint density function $\rho(P, Q)$. We start with a computation of the probability distribution function of momentum $\rho(P)$. Writing δ for the delta function at zero, we have by definition

$$\rho(P) = \left\langle \delta\Big(P - \int_{-\infty}^{\infty} dx\, u_0(x)\Big) \right\rangle = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda P} \left\langle e^{-i\lambda \int_{-\infty}^{\infty} dx u_0(x)} \right\rangle.$$

Using (3) this functional integral is Gaussian and can be simply computed to give

$$\left\langle e^{-i\lambda \int_{-\infty}^{\infty} dx u_0(x)} \right\rangle = e^{-lJ\lambda^2}.$$

The integral over λ is a Gaussian integral and we conclude that the distribution of P is also Gaussian, as could have been guessed from the very beginning, and given by

$$\rho(P) = \frac{e^{-(\frac{P}{P_0})^2}}{\sqrt{\pi} P_0}, \qquad \text{where } P_0 = 2\sqrt{lJ}. \qquad (5)$$

The joint probability distribution function can now be computed as follows. Fix q, p satisfying $q < 0, q < p$. Let x' be the first value of x for which $q(x) = q$. Define $q'(x)$ to equal $q(x)$

for $x \leq x'$ and to equal the reflection of $q(x)$ in the horizontal line $y = q$ for $x \geq x'$. Then if $Q' = \min_x q'(x)$ and $P' = q'(\infty)$ the reflection principle (see [29]), which exploits the white noise nature of u_0, states that Q', P' have the same distribution as Q, P. Then

$$\begin{aligned} \text{Prob}(Q \leq q, P \geq p) &= \text{Prob}(Q' \leq q, P' \leq 2q - p) \\ &= \text{Prob}(P' \leq 2q - p) \\ &= \int_{-\infty}^{2q-p} \frac{dz}{\sqrt{\pi} P_0} e^{-(\frac{z}{P_0})^2}. \end{aligned}$$

Differentiating in p and q we conclude that

$$\rho(p, q) = \frac{4(p - 2q)}{\sqrt{\pi} P_0^3} e^{-(\frac{p-2q}{P_0})^2}, \text{ if } \ q \leq min\{0, p\}, \tag{6}$$

and is zero for all other values of p and q.

With the help of (6) we are able to average functionals $F[u^*(t)] = F[u^*(x_i, t) : i = 1, 2, \ldots]$ with respect to the initial distribution. If however we are interested in the statistics of $u(x, t)$ at zero viscosity and large times there is still a question: is it true that in this limit $\langle F[u(t)] \rangle \sim \langle F[u^*(t)] \rangle$, or even at large times are there statistically many initials conditions such that corresponding velocity profiles haven't converged to the limiting ones? It so happens that the first alternative prevails. The detailed proofs of this fact for relevant functionals are carried out in the next section and in the appendix and are based on the following estimate on the time T^* of convergence to the limiting profile:

$$\text{Prob}(T^* > t) \leq C \left(\frac{t_c}{t} \right)^{\frac{1}{2}} \tag{7}$$

where $t_c = \sqrt{\frac{l^3}{J}}$ and C is a positive number. The proof of this estimate is fairly complicated and is allocated to the appendix. However the result itself is so important for the validity of conclusions of our paper that we decided to present here a convincing and very simple heuristic derivation of it.

By definition, $\text{Prob}(T^* < t \mid P, Q, x^*) = \text{Prob}(q \leq \Phi_t \mid P, Q, x^*)$, where Φ_t coincides for $x < x^*$ with with parabolic arc $\Phi_{1,t}$ passing through the point $(x^*, -Q)$ and touching the line $y = 0$ and with parabolic arc $\Phi_{1,t}$ passing through the point $(x^*, -Q)$ and touching the line $y = -P$ for $x > x^*$ (we used the translation invariance of the random variable T^* to set $x_0 = 0$. Consequently, $x^* \in [-l, l]$).

It is convenient to think of a Brownian walk $q(x)$ passing through $(x^*, -Q)$ as a collection of two independent walks $q^+(x)$ and $q^-(x)$ starting at this point and moving in the opposite directions in "time" x. Therefore,

$$\text{Prob}(T^* < t \mid P, Q, x^*) = \text{Prob}(q^- < \Phi_{1,t} \mid Q, x^*) \cdot \text{Prob}(q^+ < \Phi_{2,t} \mid P, Q, x^*). \tag{8}$$

To estimate, say, $\text{Prob}(q^- < \Phi_{1,t} \mid Q, x^*)$ below we note that $\text{Prob}(q^- < \Phi_{1,t} \mid Q, x^*) \geq \text{Prob}(q^- < -Q + \theta \cdot (x - x^*) \mid Q, x^*)$, where $y = -Q + \theta \cdot (x - x^*)$ is an equation for the line

tangent to the parabola $\Phi_{1,t}$ at the point $(x^*, -Q)$; $\theta = \sqrt{\frac{-2Q}{t}}$. Hence,

$$\mathrm{Prob}(q^- < \Phi_{1,\,t} \mid Q, x^*) \geq$$

$$\geq \lim_{\epsilon \to +0} \frac{\int\limits_{q(-l)=0}^{q(x^*)=-Q-\epsilon} Dq \; \Theta\Big[q < -Q + \theta \cdot (x - x^*)\Big] \; e^{-\frac{1}{2J}\int_{-l}^{x^*} \dot{q}^2 dx}}{\int\limits_{q(-l)=0}^{q(x^*)=-Q-\epsilon} Dq \; \Theta\Big[q < -Q\Big] \; e^{-\frac{1}{2J}\int_{-l}^{x^*} \dot{q}^2 dx}} \Theta\Big(0 < -Q - \theta \cdot (l + x^*)\Big), \quad (9)$$

where $\Theta[\ldots]$ is a functional step function, $\Theta(\ldots)$ - a usual one.

The functional integral in the numerator of (9) can be transformed into an integral over all pathes satisfying $q(x) \geq 0$ by a change of variables $q(x) \to q(x) - Q + \theta(x - x^*)$. (A counter part of this transformation in quantum mechanics is a Galilean transformation.) Now the functional integrals in both numerator and denumerator of (9) can be expressed in terms of Green's function of heat equation $\dot{q} = \frac{J}{2} q''$ on half a line, i.e. the antisymmetrization of Green's function of the same equation on the whole line. A simple computation shows then that

$$\mathrm{Prob}(q^- < \Phi_{1,t} \mid Q, x^*) \geq \left(1 - \sqrt{\frac{2l^2}{-Qt}}\right) \cdot \theta\left(1 - \left(\frac{2l^2}{-Qt}\right)\right). \quad (10)$$

Similar estimate holds for $\mathrm{Prob}(q^+ < \Phi_{2,t} \mid P, Q, x^*)$ if one replaces $-Q$ with $P - Q$ in the r. h. s. of (10).

Substituting these two estimates into (8) and integrating both sides of the resulting inequality w. r. t. P, Q using (6) we find that $\mathrm{Prob}(T^* < t) \geq 1 - Const\sqrt{\frac{l^2}{tP_0}}$, which is equivalent to the estimate (7) for $\mathrm{Prob}(T^* > t) = 1 - \mathrm{Prob}(T^* < t)$.

3 The statistics of the velocity field in the $\nu \to 0$, $t \to \infty$ limit.

3.1 Moments of the velocity distribution.

The aim of the present section is to compute the large t-limit of moments of the velocity distribution

$$M_n(t) = \langle u^n(0,t) \rangle, \; n = 1, 2, \ldots \quad (11)$$

Odd order moments vanish identically due to the symmetry: both Burgers equation and the initial distribution are invariant with respect to the transformation $u \to -u$, $x \to -x$. On the other hand, $M_{2k+1} \to -M_{2k+1}$ under this transformation, which implies that $M_{2k+1}(t) \equiv 0$ for $k = 1, 2, \ldots$ We concentrate therefore on the computation of the moments of even order and

assume everywhere below that n is even. We may write, using the fact that $u(x,t) = u^*(x,t)$ for $t > T^*$,

$$M_n(t) = \langle u^{*n}(0,t)\rangle + R_n(t), \tag{12}$$

where

$$R_n(t) = \left\langle \Big(u^n(0,t) - u^{*n}(0,t)\Big)\theta(T^* - t)\right\rangle \tag{13}$$

is an error term to be estimated.

The first term in the right hand side of (12) can be written in the following form:

$$\langle u^{*n}(0,t)\rangle = \int dpdq\ \rho(p,q) \int_{-L}^{L} \frac{dx_0}{2L}\ U^n(x_0,0,t,p,q) + r_n(t), \tag{14}$$

where

$$r_n(t) = \int_{-l}^{l} dx^* \int dpdq\ \rho(p,q,x^*)\Big(\int_{-L+x^*}^{-L} + \int_{L}^{L+x^*}\Big)\frac{dx_0}{2L}\ U^n(x_0,0,t,p,q) \tag{15}$$

is an error term appearing due to neglecting x^* in comparison to L and $\rho(p,q,x^*)$ is a joint probability density of P, Q and x^*. It is shown in the appendix that the error term $r_n(t)$ does not affect the asymptotics as $\nu \to 0, L \to \infty, t \to \infty$. Informally this fact can be explained by noticing that the integrand in (15) is non-zero only for velocity profiles which are "stretched" over the interval of length L and thus are exponentially improbable.

The remaining integral on the right hand side of (14) can be evaluated exactly using the explicit expressions (4) and (6) leading to the following result:

$$\langle u^{*n}(0,t)\rangle \sim \frac{\Gamma((n+3)/4)}{\sqrt{\pi}(n+1)}\frac{L(t)}{L}U(t)^n \tag{16}$$

where

$$L(t) = \sqrt{2P_0 t},\ U(t) = \frac{L(t)}{t} \tag{17}$$

are parameters, with dimensions length and velocity, which should be interpreted as the scale of turbulence and turbulent velocity correspondingly. Here we write the symbol $\sim$ to mean asymptotic equivalence in the limit as $L \to \infty$ and then $t \to \infty$.

Another computation presented in the appendix leads to the following estimate of the error term $R_n(t)$ from (12):

$$|R_n(t)| \le C_n \frac{L(t)}{L} U(t)^n \left(\frac{t_c}{t}\right)^{1/4}, \tag{18}$$

where $t_c = \sqrt{l^3/J}$ is a constant having a dimension of time, C_n is a positive constant. Comparing (18) with (16) we see that for $t >> t_c$, $|R_n(t)| << \langle u^{*n}(0,t)\rangle$, which permits us to conclude that

$$M_{2k}(t) \sim \frac{\Gamma(k/2+3/4)}{\sqrt{\pi}(2k+1)}\frac{L(t)}{L}U(t)^{2k}, \qquad k = 1,2,\ldots \tag{19}$$

It is important to stress however that coefficient C_n from (18) grows faster with n than the number factor in the r. h. s. of (19). Thus it takes a long time for a moment of high order to converge to the limiting value (19).

It follows from (19) that the energy density $E(t) \equiv \frac{1}{2}M_2(t)$ decays like $t^{-1/2}$ as $t \to \infty$. This is the result to be expected: Dissipation of energy occurs in Burgers turbulence due to shock collisions and at each separate shock. The energy of a separate shock decays as $t^{-1/2}$ and due to the absence of shock collisions in the limiting profile (4), this also gives the law of decay of total energy density. This argument is due to J. M. Burgers, see [11].

We will also see below that the statistics of the velocity field in our model is self-similar with the scales of length and velocity given by (17). These scales depend on time exactly as their counterparts in Kida's model. The statistics of the velocity field in our case are however different from that of Kida[1]. Thus we conclude that the self-similarity alone does not determine the large time asymptotics of the statistics of velocity field in DBT. Note also that $E(t)$ decays in time, showing the presence of a dissipation anomaly in the model: the rate of energy dissipation does not vanish but converges to a finite non-zero limit when the viscosity ν approaches zero.

3.2 The probability distribution function of velocities.

In this section we will concern ourselves with computing the probability distribution function (PDF) of velocities given by

$$P(u,t) \equiv \mathrm{Prob}(u(0,t) > u) = \langle \theta(u(0,t) - u) \rangle \,. \tag{20}$$

Reasoning exactly as in the previous section we find that

$$P(u,t) = \langle \theta(u^*(0,t) - u) \rangle + R(u,t), \tag{21}$$

where

$$R(u,t) = \left\langle \left(\theta\Big(u(0,t) - u \Big) - \theta\Big(u^*(0,t) - u \Big) \right) \theta(t^* - t) \right\rangle \tag{22}$$

is an error due to the replacement $u \to u^*$;

$$\langle \theta(u^*(0,t) - u) \rangle = \\ = \theta(-u) + \int dpdq\, \rho(p,q) \int_{-L}^{L} \frac{dx_0}{2L} \Big(\theta(U(x_0, 0, t, p, q) - u) - \theta(-u) \Big) + r(u,t), \tag{23}$$

where $r(u,t)$ is an error due neglecting x^* in comparison with L:

$$r(u,t) = \\ = \int_{-l}^{l} dx^* \int dpdq\, \rho(p,q,x^*) \Big(\int_{-L+x^*}^{-L} + \int_{L}^{L+x^*} \Big) \frac{dx_0}{2L} \left(\theta(U(x_0,0,t,p,q) - u) - \theta(-u) \right) . \tag{24}$$

[1] There exists no complete solution of Kida's model. Yet the answers which can be obtained within Kida's model are different from their counterparts in our model.

The reason that the term $\theta(-u)$ is added and subtracted is that, due to the averaging of the position of the initial condition over the block $[-L, L]$, the velocity is typically zero and so the PDF is an $O(L^{-1})$ perturbation to $\theta(-u)$.

An estimate of $r(u,t)$ similar to that of the term $r_n(t)$ in section (3.1) shows that $r(u,t)$ does not affect the final asymptotics. An exact calculation using the known density $\rho(p,q)$ for the other terms on the right hand side of (23) leads to

$$\langle \theta(u^*(0,t)-u)\rangle \sim \theta(-\bar{u}) + \frac{L(t)}{L}\int_{\bar{u}^2}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2}\Big(\sqrt{\alpha}-|\bar{u}|\Big)\mathrm{sgn}(\bar{u}) \tag{25}$$

where $\bar{u} = u/U(t)$. A computation performed in the appendix shows that

$$|R(u,t)| \leq C\frac{L(t)}{L}(\frac{t_c}{t})^{1/4} \tag{26}$$

where C is a positive constant. Comparing (26) with (25) we see that for $t >> t_c$ we have $\langle \theta(u^*(0,t)-u)\rangle >> |R(u,t)|$, with the last inequality being pointwise in $\bar{u}$ rather than uniform. We conclude that

$$P(U(t)\bar{u},t) \sim \theta(-\bar{u}) + \frac{L(t)}{L}\int_{\bar{u}^2}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2}\Big(\sqrt{\alpha}-|\bar{u}|\Big)\mathrm{sgn}(\bar{u}) \tag{27}$$

If in particular $|\bar{u}| \to \infty$ this simplifies to

$$P(U(t)\bar{u},t) \sim \theta(-\bar{u}) + \frac{1}{8\sqrt{\pi}}\mathrm{sgn}(\bar{u})\frac{L(t)}{L}\frac{e^{-\bar{u}^4}}{\bar{u}^5}. \tag{28}$$

Note that the answer (27) for $P(u,t)$ is self-similar with $U(t)$ playing the role of the integral velocity scale. Note also that the form of $P(u,t)$ is not Gaussian. This confirms the non-triviality of our model: the output (the strongly non-Gaussian statistics of the velocity field in the limit of small viscosity and large time) is not the same as the input (a trivial Gaussian distribution of the initial velocity field). This non-triviality will be re-emphasised in the consequent sections where it will be shown that the limiting statistics of the velocity field is intermittent.

Finally we would like to make the following technical comment. Of course, the moments of the distribution (27) are exactly those given by (19). We could therefore try to compute the distribution (20) first and then argue that the moments of this asymptotic distribution coincide with the asymptotics of the moments of the actual distribution. Unfortunately the analysis of error terms within this approach becomes very involved. For this reason we have two separate computations, the asymptotics of the moments of the velocity distribution and the asymptotics of the velocity distribution itself.

3.3 Velocity structure functions.

Now we will turn to the two-point statistics of the velocity field and compute asymptotics for the velocity structure functions given by

$$S_n(y,t) = \left\langle \Big(u(y,t) - u(0,t) \Big)^n \right\rangle \quad n = 1, 2, \dots \tag{29}$$

We find as in the previous subsections that

$$S_n(y,t) = \left\langle \Big(u^*(y,t) - u^*(0,t) \Big)^n \right\rangle + R_n(y,t), \tag{30}$$

where $R_n(y,t)$ accounts for the error due to the replacement of u with u^*. As shown in the appendix this error can be estimated as follows: for t such that $L(t) \geq y$

$$|R_n(y,t)| \leq C_n U^n(t) \frac{y}{L} \left(\frac{t_c}{t} \right)^{1/4}. \tag{31}$$

We express the first term in (30) as

$$\left\langle \Big(u^*(y,t) - u^*(0,t) \Big)^n \right\rangle =$$

$$= \int_{-L}^{L} \frac{dx_0}{2L} \int dpdq\, \rho(p,q) \Big(U(x_0, y, t, p, q) - U(x_0, 0, t, p, q) \Big)^n + r_n(y,t)$$

where $r_n(y,t)$ accounts for an error arising due to neglecting x^* in comparison with L. Again it can be shown that the term $r_n(y,t)$ does not contribute to the asymptotics. Now a direct computation using the density $\rho(p,q)$ shows, for $y \geq 0$, that

$$\left\langle \Big(u^*(y,t) - u^*(0,t) \Big)^n \right\rangle \sim (-1)^n \frac{1}{\sqrt{\pi}} \Gamma\Big(\frac{n+2}{4} \Big) \frac{L(t)}{L} U^n(t) \bar{y} + O(\bar{y}^2), \ n = 2, 3, \dots$$

where $\bar{y} = \frac{y}{L(t)}$. In addition $S_1(y,t) \sim 0$, which confirms the restoration of translation invariance in the large L limit. Comparing this with (31) we see that the asymptotics of the velocity structure functions is given, for fixed $\bar{y} \leq 1$, by

$$S_n(L(t)\bar{y}, t) \sim (-1)^n \frac{1}{\sqrt{\pi}} \Gamma(\frac{n+2}{4}) \frac{L(t)}{L} U^n(t) \bar{y} + O(\bar{y}^2) \quad n = 2, 3, \dots \tag{32}$$

It has been assumed in our computations that $\bar{y} \geq 0$. Extending (32) to negative y by the symmetry $y \to -y$, $u \to u$, we see that $S_{2k}(L(t)\bar{y}, t)$ is proportional to $|\bar{y}|$ and $S_{2k+1}(L(t)\bar{y}, t)$ is proportional to $\bar{y}$ for $k \geq 1$ and $|\bar{y}| << 1$.

Thus the velocity structure functions of the problem exhibit in the inertial range the extreme anomalous (non-Kolmogorov) scaling which is typical for Burgers turbulence in general and is due to the presence of shocks in the limiting velocity profile. The Burgers anomalous scaling is well known from heuristic arguments (see e.g. [15], [7], [9]). In our case however it has been derived as a part of the complete solution of the problem.

3.4 The probability distribution function of velocity differences.

Here we will compute the PDF for velocity differences

$$P(u,y,t) = \mathrm{Prob}\Big(u > \Delta u(y,t)\Big) = \Big\langle \theta\Big(u - \Delta u(y,t)\Big)\Big\rangle, \tag{33}$$

where $\Delta u(y,t) = u(y/2,t) - u(-y/2,t)$ and $y \geq 0$. Definition (33) is tailored for the study of negative velocity differences and we consider only the case $u < 0$. Negative differences are the interesting case since they occur when the velocities are evaluated either side of a shock. A lengthy but straightforward computation shows that for fixed $\bar{u} < 0$, $\bar{y} > 0$

$$P(U(t)\bar{u}, L(t)\bar{y}, t) \ \sim$$

$$\sim \ 2\frac{L(t)}{L}\Big(\int_{\bar{u}^2}^{(\bar{y}-\bar{u})^2} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2}\Big(\sqrt{\alpha}+\bar{u}\Big) + \bar{y}\int_{(\bar{y}-\bar{u})^2}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2}\Big) + R(\bar{u},\bar{y},t) \tag{34}$$

where, as shown in the appendix,

$$|R(\bar{u},\bar{y},t)| \leq C\frac{L(t)}{L}\frac{\bar{y}}{\bar{y}+|\bar{u}|}\left(\frac{t_c}{t}\right)^{1/4}. \tag{35}$$

Due to the presence of extra factor of $(\frac{t_c}{t})^{1/4}$ decaying with time, $R(u,y,t)$ becomes small compared to the first term in the right hand side of (34), given that $\bar{u}, \bar{y}$ fixed.

It is easy to analyse (34) in the following limiting cases. We suppose that $\bar{y} << 1$. If $|\bar{u}| << 1$, then

$$P(U(t)\bar{u}, L(t)\bar{y}, t) \sim \frac{L(t)\bar{y}}{L}\left(1 - \frac{\sqrt{\pi}}{2}\bar{y} - \frac{\sqrt{\pi}}{2}|\bar{u}|^2 + O(\bar{y}^2) + O(|\bar{u}|^3)\right). \tag{36}$$

If $1 << |\bar{u}| << \bar{y}^{-1/3}$ then

$$P(U(t)\bar{u}, L(t)\bar{y}, t) \sim \frac{L(t)\bar{y}}{\sqrt{\pi}L}\frac{1}{|\bar{u}|^2}e^{-|\bar{u}|^4}\left(1 + O(\frac{1}{|\bar{u}|^4}) + O(\bar{y}|\bar{u}|^3)\right) \tag{37}$$

If $|\bar{u}| >> \bar{y}^{-1/3}$ then

$$P(U(t)\bar{u}, L(t)\bar{y}, t) \sim \frac{L(t)}{4\sqrt{\pi}L}\frac{1}{|\bar{u}|^5}e^{-|\bar{u}|^4}\left(1 + O(\frac{1}{|\bar{u}|^8})\right) \tag{38}$$

To summarize, for negative u, $P(u,y,t)$ decays algebraically for $|u| << U(t)$ and super exponentially for $|u| >> U(t)$. Moreover, $P(u,y,t) \sim O(y)$ if $1 << |\bar{u}| << \bar{y}^{-1/3}$ and doesn't depend on y if $1 << |\bar{u}| << \bar{y}^{-1/3}$. This information alone enables one to conclude that velocity structure functions of sufficiently high order exhibit anomalous scaling. In addition we observe a crossover between regimes (37) and (38). This crossover is actually responsible for the presence of many scales in description of the statistics of velocity field and the absence of the universal inertial range in Burgers turbulence. We refer reader to section 4 for a detailed discussion of this point.

3.5 The multi-time statistics of the velocity field.

The simplicity of our model allows us to compute the correlation between values of the velocity field at different moments of time. Let

$$T_n(\tau,t)=\left\langle\Big(u(0,t+\tau)-u(0,t)\Big)^n\right\rangle \tag{39}$$

be the velocity structure functions corresponding to the same point at space but different moments of time. We write (39) in the already familiar form

$$T_n(\tau,t)=\left\langle\Big(u^*(0,t+\tau)-u^*(0,t)\Big)^n\right\rangle+R_n(\tau,t) \tag{40}$$

with $R_n(\tau)$ accounting for an error due to the replacement of u with u^*. An estimate in the appendix shows that

$$|R_n(\tau,t)|\leq C_nU^n(t)\frac{\tau U(t)}{L}\left(\frac{t_c}{t}\right)^{1/4}. \tag{41}$$

The computation of the first term in the right hand side of (40) is very close to the computation performed in previous sections and leads to

$$\begin{aligned} T_n(\tau,t) &\sim (-1)^n\frac{\Gamma(\frac{n+3}{4})}{2\sqrt{\pi}}U^n(t)\frac{U(t)\tau}{L}+R_n(\tau,t)\\ &\sim (-1)^n\frac{\Gamma(\frac{n+3}{4})}{2\sqrt{\pi}}U^n(t)\frac{U(t)\tau}{L},\quad n=2,3,\ldots \end{aligned} \tag{42}$$

We therefore conclude that the time-like structure functions exhibit in Burgers turbulence the extreme anomalous scaling in τ given by $T_n(\tau,t)\sim\tau, n=2,3,\ldots$. Comparing this with the expression (32) for the space-like structure functions, we see that

$$S_n(y,t)=T_n(\tau,t),\qquad n=2,3,\ldots \tag{43}$$

at $y=C(n)U(t)\tau$, given that $y<<L(t)$ and $\tau<<t$. The identity (43) means that "isotropic" Taylor conjecture stating the equivalence of the space-like and time-like statistics in isotropic turbulence at small scales, becomes a theorem for our model of Burgers turbulence. The similar observation was also independently made in [8] in the context Burgers turbulence generated by correlated Gaussian initial conditions.

Let us finally note that if one wishes to compare $T_n(y,t)$ with $S_n(\tau,t)$ at arbitrarily high orders n, the condition of applicability of relation (43) has to be changed to $y<<L_n(t)$, where $L_n(t)$ is correlation length associated with n-th order structure function introduced in section 4.2. For $n>>1$, $L_n(t)\sim L(t)/n^{3/4}$, see 46 below.

3.6 One-shock approximation.

We wish to show that all of the results obtained in the previous section can be easily obtained from heuristic arguments given the knowledge of the probability density of a velocity jump

at a shock. In our case the latter is easy to compute: a simple computation which uses the knowledge of the limiting velocity profile (4) and the density $\rho(p,q)$ gives

$$\rho(\mu) \equiv \Big\langle \delta\Big(\mu - \sqrt{2(P-Q)/t}\Big)\Big\rangle = \frac{4}{\sqrt{\pi}U(t)}\bar{\mu}e^{-\bar{\mu}^4}, \tag{44}$$

where μ is a velocity jump at the (right) shock, $\bar{\mu} = \frac{\mu}{U(t)}$. The probability density of the velocity jump at the left shock has exactly the same form, so we will be referring to (44) as the probability density of the velocity jump at a shock.

Now let us *assume*: Firstly that the large-t statistics of u are approximated by that of u^*; secondly that a one-shock approximation is valid, i.e. that one can disregard in the analysis the contributions coming from configurations with shocks separated by distances much less than the average separation $L(t)$.

To derive $P(u,y,t)$ for $u<0,\ y<<L(t)$ using these assumptions note that $u(y,t)-u(0,t)$ can be negative only if there is a shock at some point in $[0,y]$. If the right hand shock lies at $x\in[0,y]$ then $u(y,t)-u(0,t) = -\mu + x/t$. A similar formula holds if the left hand shock lies in $[0,y]$. So neglecting the contribution from the configurations with 2 shocks inside the interval $[0,y]$, we see that

$$\mathrm{Prob}\Big(u(y,t)-u(0,t)<u\Big) \approx 2\int_0^y \frac{dx}{2L}\,\mathrm{Prob}(\text{Size of Jump} > \frac{x}{t} - u).$$

This can be easily computed using the density of the shock jump (44) giving

$$\mathrm{Prob}\Big(u(y,t)-u(0,t)<y\Big) \approx \frac{2L(t)}{L}\Big(\int_{\bar{u}^2}^{(\bar{y}-\bar{u})^2}\frac{d\alpha}{\sqrt{\pi}}e^{-\alpha^2}(\sqrt{\alpha}+\bar{u}) + \bar{y}\int_{(\bar{y}-\bar{u})^2}^{\infty}\frac{d\alpha}{\sqrt{\pi}}e^{-\alpha^2}\Big), \tag{45}$$

which coincides with the exact answer (34).

With the knowledge of the PDF of velocity difference we can compute velocity structure functions, thus moments of velocities, thus the PDF of velocities. In other words all of the results of the previous section concerning single time statistics of the velocity field can be obtained using a one-shock approximation.

Moreover, the τ-dependence of the time-like structure functions (42) is also entirely due to the one-shock effects: if $n=2,3,\ldots$ and $\tau << t$, then the main contribution to $T_n(\tau,t)$ comes from the configurations with a shock passing through $x=0$ between the moments of time t and $t+\tau$. A shock with velocity jump μ travels a distance approximately $\mu\tau/2$ over the interval $[t,t+\tau]$. Therefore,

$$\begin{aligned} T_n(\tau) &\approx \Big\langle (-\mu)^n \chi\Big(\text{Shock passed through 0 during } [t,t+\tau]\Big)\Big\rangle \\ &\approx \langle(-\mu)^n\frac{\mu\tau}{2L}\rangle. \end{aligned}$$

Computing this average using the PDF of shock strength (44) we arrive exactly at (42), which again shows that one-shock approximation is asymptotically exact.

These calculations support the following statement about decaying Burgers turbulence: all one needs to know in order to describe the statistics of the velocity field at scales much less than the average distance between shocks is the *one*-point PDF of shock velocity and strength (or just shock strength if the correlation functions which we're trying to compute are Galilean-invariant). Thus the problem is much simpler than one might have thought: recall for example that exact formulae expressing velocity correlation functions in terms of the statistics of shocks are such ([11], [23]) that one seemingly needs to know the n-point joint PDF of shock strengths in order to compute n-th order correlation function.

The rigorous proof of the above statement together with estimates on the errors of one-shock approximation will make DBT analytically tractable for a wide class of initial conditions as the great deal is known about the one-point function of shock strength, see e. g. [11], [23], [32], [3], [4].

Is there a universal technique for the computation of the one-point PDF of the shock strength? It has been known since Burgers [11], but never really exploited, that shocks behave (almost) as a system of sticky particles. One might try therefore to extract the information about one-point PDF of shock characteristics by studying the kinetics of this system, for example, by analyzing the Smoluchowski-Bogoluibov chain of equations for one-point, two-point, ... PDF's of shocks.

3.7 On multiscaling in Burgers turbulence.

In statistical physics the term "multiscaling", instead of "anomalous scaling", is used to stress an inherently multiscale nature of a system exhibiting anomalous scaling of correlation functions. Burgers turbulence is no exception. In this section we will show that the crossover between the tails (37) and (38) of the PDF for velocity differences is actually a reflection of the presence of many correlation lengths in the problem, which in turn is a consequence of the anomalous scaling of correlation functions and, ultimately, the intermittency of the velocity field in Burgers turbulence.

Let $n >> 1$ be a large even positive integer. We know from (32) that as $\bar{y}$ approaches zero

$$S_n(L(t)\bar{y},t) \approx \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{n+2}{4}\right)\frac{L(t)}{L}U^n(t)\bar{y}.$$

For large $\bar{y}$ however one expects the quantities $u(L(t)\bar{y},t)$ and $u(0,t)$ to become independent. When this happens we have

$$\begin{aligned} S_n(L(t)\bar{y},t) &= \langle (u(L(t)y,t)-u(0,t))^n\rangle \\ &\sim \langle u^n(L(t)y,t)\rangle + \langle u^n(0,t)\rangle \\ &\sim 2M_n = 2\frac{\Gamma(\frac{n+3}{4})}{\sqrt{\pi}(n+1)}\frac{L(t)}{L}U(t)^n. \end{aligned}$$

Here the cross terms in expanding the nth power are, using the independence, of order $O(L^{-2})$. The region in between these two formulae for large and small y marks the correlation length

for the nth moments. If we assume there is a simple crossover then we can locate the scale at which it occurs by equating the expressions for large $\bar{y}$ and small $\bar{y}$. These become equal, i.e. $S_n(L(t)\bar{y}, t) \approx 2M_n$, at the value $n^{-3/4}$ and so the correlation length for the n-th structure function is

$$L_n \sim \frac{L(t)}{n^{3/4}}, \quad n >> 1 \tag{46}$$

and this shows the presence of many scales in our problem.

To show how this multiscaling is related to the crossover between the asymptotic regimes (37) and (38) we shall use the PDF for velocity differences to compute $S_n(y,t)$ for n positive and large. Writing $S_n(L(t)\bar{y}, t)$ as an integral against the PDF of $\Delta u(L(t)\bar{y}, t)$ and treating n as a large parameter we see that the integral is dominated by values of $|\bar{u}|$ coming from the neighbourhood of the negative critical point of the function

$$F(u) = |\bar{u}|^n \exp(-|\bar{u}|^4)$$

namely near $\bar{u}_c = -n^{1/4}$. Note this value is much less than -1 for $n >> 1$ and so we may neglect the part of the integral that uses the PDF in the form (36) and also neglect positive values of $\Delta u(y)$. Now, if in addition $|\bar{u}_c| << \bar{y}^{-1/3}$, we have to use asymptotics (37) to evaluate the contribution from the critical point, which yields $S_n(L(t)\bar{y}, t) \approx C\bar{y}$. If $|\bar{u}_c| >> \bar{y}^{-1/3}$ we have to use asymptotics (38) in our computations, which gives $S_n(L(t)\bar{y}, t) \approx$ Constant. The crossover between these two answers corresponds to the crossover between the asymptotics (36) and (37) and occurs when $\bar{y} = |\bar{u}_c|^3 = n^{-3/4}$, exactly as in our computed correlation length L_n for the n-th structure function.

It remains to remark that multiscaling, and consequently a PDF for velocity differences which has a crossover between a regime scaling like y and one that is independent of y, should be a general feature of DBT regardless of the initial distribution. All related questions concerning other statistics can be studied in more general situations, if one assumes a one-shock approximation is valid, by using the information about the tails of the one-point PDF of shock strength obtained in [32], [3], [4].

It is worth noting that the presence of the multitude of correlation lengths in Burgers turbulence was understood long ago by Robert Kraichnan, [25], and rediscovered within the instanton approach to the forced Burgers turbulence, [12]. It is also worth stressing that in models of chaotic systems which do not account for the effects of intermittency, there is always a single universal correlation length. A good example is served by random matrix models, see [26] for a review.

Finally, let us remark that if we define the integral scale as the scale of scaling behaviour of correlation functions, we must immediately conclude that there is no such unique scale, there is rather a family of them parameterized by the order of correlation function. In other words, the notion of the integral scale becomes *local*, and the notion of the universal inertial range disappears. (See also [14] for the general discussion about the multitude of *dissipative* scales based on a multifractal models). This should be a general feature of all intermittent turbulent systems, for instance, Navier-Stokes turbulence.

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5 Note added in proof.

We are grateful to the referee of our paper who drew our attention to a recent preprint by L. Frachebourg and Ph. A. Martin, [13], in which the study of the model of decaying Burgers turbulence initiated by white noise initial conditions (without compactness assumption) has been effectively completed. This model was originally considered by Burgers himself about forty years ago but complexity of analysis prevented him from obtaining explicit answers for anything but the two- and three-point correlation functions of velocity field. Now most of the questions about the statistics of velocity field in Burgers' model can be effectively resolved using the integral representation of the Green's function of a diffusion equation in the (x,t)-domain with parabolic boundary derived in the above mentioned paper.

6 Appendix.

In order to bound the various error terms in section 3 we will need to bound the size of the true solution u, the asymptotic solution u^* and the size of their supports (i.e. the interval on which they are non-zero). We use details from the method of construction of the vanishing viscosity solution as descibed in [20] and recalled in section 2.

Suppose that initial velocity profile is supported in the interval $(x_0 - l, x_0 + l)$. The rightmost (respectively leftmost) parabola in the chain of parabolic arcs built on the initial potential will always lie to the left (respectively right) of the parabola with the same curvature that passes through the point $(x_0 + l, -Q)$ (respectively $(x_0 - l, -Q)$) and assumes minimial value equal to $-P$ (respectively 0). This immediately implies that both u and u^* are supported in the interval $[y_*, y^*]$ where

$$y_* = x_0 - l - \sqrt{-2tQ}, \quad y^* = x_0 + l + \sqrt{2t(P-Q)}. \tag{47}$$

Using the fact that both u and u^* vanish at the point within $[x_0 - l, x_0 + l]$ at which $q(x)$

[2]Who asked the very useful, perhaps rhetorical, question "Why study white noise Burgers turbulence at all?"

achieves its global minimum, we also find that $|u|$ and $|u^*|$ are bounded by $u_{\max}$ where

$$\begin{aligned} u_{\max} &= \max\{(y^* - (x_0 - l))/t, ((x_0 + l) - y_*)/t\} \\ &= \max\{(2l + \sqrt{2t(P - Q)})/t, (2l + \sqrt{-2tQ})/t\}. \end{aligned} \tag{48}$$

Estimates (47), (48) and the bound (7) will be used to estimate all relevant error terms in section 3. The careful analysis of these error terms leads to a better understanding for when the asymptotics for various statistics start to hold.

6.1 Proof of the estimate (18).

Applying the above estimates to the error term (18) we obtain

$$\begin{aligned} |R_n(t)| &= \langle |u^n(0,t) - u^{*n}(0,t)| \theta(T^* - t)\rangle \\ &= \langle |u^n(0,t) - u^{*n}(0,t)| \chi_{[y_*, y^*]}(0)\; \theta(T^* - t)\rangle \\ &\leq 2\langle u^n_{\max} \chi_{[y_*, y^*]}(0)\; \theta(T^* - t)\rangle \\ &\leq \frac{2}{L}\langle (y^* - y_*) u^n_{\max} \theta(T^* - t)\rangle \quad \text{(averaging over } x_0\text{)} \\ &\leq \frac{2}{L}\langle (y^* - y_*)^2 u^{2n}_{\max}\rangle^{1/2} \langle \theta(T^* - t)\rangle^{1/2} \quad \text{(Cauchy-Schwartz)} \\ &\leq C_n \frac{L(t)}{L} U^n(t) (\frac{t^*}{t})^{1/4}, \end{aligned}$$

where the last inequality uses the the estimate (7) and an explicit calculation using $\rho(p, q)$. Comparing the first and the last entries of the presented chain of inequalities we obtain a proof of (18).

6.2 Proof of the estimate on $r_n(t)$ from section (3.1).

We may bound $r_n(t)$ as follows:

$$\begin{aligned} |r_n(t)| &\leq \int_{-l}^{l} dx^* \int dpdq\; \rho(p, q, x^*) (\int_{-L}^{-L+x^*} + \int_{L}^{L+x^*}) \frac{dx_0}{2L} |U(x_0, 0, t, p, q)|^n \\ &\leq \int dpdq\; \rho(p, q) (\int_{-L-l}^{-L+l} + \int_{L-l}^{L+l}) \frac{dx_0}{2L} |U(x_0, 0, t, p, q)|^n \\ &\leq \frac{2l}{L} \left(\frac{L + l}{L}\right)^n \int dpdq\; \rho(p, q) \theta(\sqrt{-2Qt} - (L - l)) U^n(t) \\ &\leq \frac{2l}{L} \left(\frac{L + l}{L}\right)^n \left(\frac{L(t)}{L - l}\right)^2 \exp\left(-\left(\frac{L - l}{L(t)}\right)^4\right) U^n(t) \end{aligned}$$

where the last inequality follows by an explicit calculation using $\rho(p, q)$. This is exponentially small in L and so does not affect the asymptotics which take the limit $L \to \infty$ first and preserve only the $O(L^{-1})$ terms. A similar argument controls similar error terms of this form for the other statistics considered.

6.3 Proof of the estimate (26).

The proof of (26) is similar to that of (18):

$$
\begin{aligned}
|R(u,t)| &\le \; = \langle|\theta(u(0,t)-u)-\theta(u^*(0,t)-u)|\chi_{[y_*,y^*]}(0)\theta(T^*-t)\rangle \\
&\le 2\langle\chi_{[y_*,y^*]}(0)\theta(T^*-t)\rangle \\
&\le \frac{1}{L}\langle(y^*-y_*)\theta(T^*-t)\rangle \\
&\le \frac{1}{L}\langle(y^*-y_*)^2\rangle^{1/2}\langle\theta(T^*-t)\rangle^{1/2} \\
&\le C\frac{L(t)}{L}(\frac{t_c}{t})^{1/4}.
\end{aligned}
$$

6.4 Proof of the estimate (31).

We can split this error term into two via

$$
|R_n(y,t)| \le 2^n\langle\Delta u(y,t)^n\theta(T^*-t)\rangle + 2^n\langle\Delta u^*(y,t)^n\theta(T^*-t)\rangle, \tag{49}
$$

where $\Delta u(y) = u(y/2,t) - u(-y/2,t)$. We show how to bound the first of these terms, the other being entirely similar. The vanishing viscosity solution u takes the form, within its support, of a line with slope $1/t$ plus a series of downward jumps. So we may define $F(x,t)$ to be a non increasing piecewise constant function so that, for x in the support of u,

$$
u(y,t) = \frac{y-x_0}{t} + F(y-x_0,t).
$$

It is easy to see that $|\Delta F(y,t)| = |F(y/2,t) - F(-y/2,t)| \le 2u_{\max}$. Also $|\Delta u(y,t)| \le |y/t| + |\Delta F(y-x_0,t)|$ whenever one of the points $y/2$ or $-y/2$ is in the support of u. So we bound the first term on the right hand side of (49) by

$$
\begin{aligned}
&\langle(|y/t| + |\Delta F(y-x_0)|)^n\chi_{[y_*-(y/2),y^*+(y/2)]}(0)\theta(T^*-t)\rangle \\
\le\; & 2^n\langle|\Delta F(y-x_0)|^n\theta(T^*-t)\rangle + 2^n\langle|y/t|^n\chi_{[y_*-(y/2),y^*+(y/2)]}(0)\theta(T^*-t)\rangle.
\end{aligned} \tag{50}
$$

The first term on the right hand side of (50) can be bounded by averaging over x_0 first and using

$$
\begin{aligned}
\int_{-L}^{L}\frac{dx_0}{2L}|\Delta F(y-x_0,t)|^n &\le (2u_{\max})^{n-1}\int_{-L}^{L}\frac{dx_0}{2L}|\Delta F(y-x_0,t)| \\
&\le (2u_{\max})^{n-1}\frac{yu_{\max}}{L},
\end{aligned}
$$

using in the last inequality the fact that F is decreasing and bounded by $2u_{\max}$. Substituting into (50) one can take the further averaging as for previous error bounds. By taking t large enough that $L(t) \ge y$ and combining the various terms one arrives at the desired error bound.

6.5 The proof of the estimate (35).

The proof of this estimate is similar to that of (31). Noting that $\Delta u(y,t) = \Delta F(y - x_0, t) + (y/t)$ we may write

$$\begin{aligned} \int \frac{dx_0}{2L}\theta(u - \Delta u(y,t)) &= \int \frac{dx_0}{2L}\theta(|\Delta F(y-x_0)| - (y/t) - |u|) \\ &\le \int \frac{dx_0}{2L}\frac{|\Delta F(y-x_0)|}{(y/t)+|u|} \\ &\le \frac{1}{2L}\frac{2yu_{\max}}{(y/t)+|u|)}. \end{aligned}$$

A similar estimate holds for $\Delta u^*(y,t)$. Hence

$$\begin{aligned} |R(u,y,t)| &\le \langle(\theta(u - \Delta u(y,t)) + \theta(u - \Delta u^*(y,t))\,\theta(T^* - t)\rangle \\ &\le \frac{1}{L}\frac{2y}{(y/t)+|u|)}\langle u_{\max}\theta(T^* - t)\rangle \\ &\le \frac{L(t)}{L}\frac{\bar{y}}{\bar{y}+|\bar{u}|}\left(\frac{t_c}{t}\right)^{1/4}. \end{aligned}$$

6.6 Proof of the estimate (41).

The proof of this estimate is similar to that of (35). The key change is to obtain a bound for

$$\int \frac{dx_0}{2L}|F(y-x_0,t) - F(y-x_0,t+\tau)|. \tag{51}$$

The piecewise constant profile $F(y,t)$ consists of a series of shocks which may travel forwards or backwards but move with a maximum speed $u_{\max}$. The total height of the shocks is also bounded by $u_{\max}$. So the integral (51) can be bounded by $u_{\max}^2\tau/2L$. The possibility of infinitely many shocks, or the merging of shocks between times t and $t+\tau$, does not affect this upper bound.

6.7 The proof of the estimate (7).

The construction of the two shock profile uses two parabolas that pass through the graph of the Brownian motion $-q(x) = -\int_{-\infty}^{x} u_0(z)dz$ at its point of maximum. Below is a lemma about the behavior of a Brownian path near its maximum.

Lemma 1 *Let $(B_t : 0 \le t \le 1)$ be a standard Brownian motion started at zero. Define*

$$M = \sup_{t\in[0,1]} B_t, \quad \Sigma = \inf\{t : B_t = M\}.$$

We consider the pieces of the path (B_t) either side of its maximum by defining

$$X_t = M - B_{\Sigma - t} \textit{ for } t \in [0, \Sigma], \quad \bar{X}_t = M - B_{\Sigma + t} \textit{ for } t \in [0, 1-\Sigma].$$

Define the slopes of two lines that pass through the maximum and lie above the path by

$$\Theta = \inf\{X_t/t : 0 < t \leq \Sigma\}, \quad \bar{\Theta} = \inf\{\bar{X}_t/t : 0 < t \leq 1 - \Sigma\}.$$

a) *The triples $(M, \Sigma, (X_t : t \leq \Sigma))$ and $(M - B_1, 1 - \Sigma, (\bar{X}_t : t \leq 1 - \Sigma))$ are identically distributed.*

b) *The law of (M, Σ) is given by*

$$P(M \in dm, \Sigma \in d\sigma) = \frac{m\sigma^{-1}}{\pi(\sigma(1-\sigma))^{1/2}} \exp(-m^2/2\sigma) dm \, d\sigma.$$

c) *Conditional on $M \in dm, \Sigma \in d\sigma$ the path $(X_t : t \leq \sigma)$ satisfies $X_0 = 0$ and solves the stochastic differential equation, driven by a Brownian motion (W_t),*

$$dX_t = f(t, X_t)dt + dW_t, \quad \textit{where } f(t,x) = \tfrac{m-x}{\sigma - t} + \tfrac{2m}{\sigma - t}(\exp(\tfrac{2mx}{\sigma - t}) - 1)^{-1}. \tag{52}$$

d) *For (X_t) that solves (52) we have the estimate*

$$P(\Theta \leq \theta) \leq C\theta(m + \sigma m^{-1}) + I(m \leq \theta\sigma).$$

We delay the proof of this lemma until the end of this appendix and first use it to prove the estimate (7) on the tail $P(T^* \geq t)$ of the time T^* at which the two shock profile is obtained. The construction of the two shock profile uses the function $q(x) = \int_\infty^x u_0(z)dz$, its global minimum Q and the position $x_0 + x^*$ at which the minimum is attained. Two parabolas of the form $\pi(z) = (z-x)^2/2t$ (and $\bar{\pi}(z) = P + (z - \bar{x})^2/2t$) are constructed to pass through the point $(x_0 + x^*, -Q)$. The slopes of the parabolas at the point $x_0 + x^*$ are $(-2Q/t)^{1/2}$ (respectively $(2(P-Q)/t)^{1/2}$). Let T (respectively $\bar{T}$) be the smallest time t at which the parabola π (respectively $\bar{\pi}$) lies above the graph of $-q(x)$. Then the two shock profile is attained for times $t \geq T^* = \max\{T, \bar{T}\}$.

To apply the lemma we must rescale to obtain a standard Brownian path of length one. Set $B_t = -(2lJ)^{-1/2} q(x_0 - l + 2lt)$ for $t \in [0,1]$. Then (B_t) is a standard Brownian motion and its maximum M takes the value $-Q/(2lJ)^{1/2}$. The construction of the parabola π (respectively $\bar{\pi}$) show that if $\Theta \leq (-4lQ/tJ)^{1/2}$ then $t \leq T$ (respectively if $\bar{\Theta} \leq ((4l(P - Q)/tJ)^{1/2}$ then $t \leq \bar{T}$). Part a) of the lemma shows that both of these events have the same probability. So, applying part d) of the lemma,

$$\begin{array}{rcl} P(T^* \geq t) & \leq & 2P(\Theta \leq (-4lQ/tJ)^{1/2}) \\ & = & 2P(\Theta \leq 2^{5/4} M^{1/2} t^{-1/2} l^{3/4} J^{-1/4}) \\ & \leq & Ct^{-1/2} l^{3/4} J^{-1/4} E(M^{1/2}(M + \Sigma M^{-1})) \\ +2P(M \leq 2^{5/4} M^{1/2} t^{-1/2} l^{3/4} J^{-1/4} \Sigma) & & \\ & \leq & Ct^{-1/2} l^{3/4} J^{-1/4} \end{array}$$

using Markov's inequality in the last inequality and the exact distribution of (M, Σ) in part b) of the lemma. This completes the proof of (7) and it remains to describe the proof of the lemma.

Part a) of the lemma follows from the symmetry of the problem with respect to the time reversal $t \to 1-t$. The distribution of (M, Σ) is well known and may be obtained for example by exploiting the reflection principle. Conditional on $M \in dm, \Sigma \in \sigma$ the path (X_t) becomes a Brownian bridge, taking the value zero at time zero and the value m at time σ, that is conditioned to never take negative values. The equation describing the evolution can then be obtained using an h-transform as in Rogers and Williams [30] section 4.23.

We first sketch the idea for estimating $P(\Theta \leq \theta) = P(X_s < \theta s \text{ for some } s \leq \sigma)$. The drift $f(t,x)$ in equation (52) is approximately $1/x$ for small t and x. If this approximation were exact the process (X_t) would satisfy $dX = X^{-1}dt + dW$ which is uniquely solved by the three dimensional Bessel process (the radius of a three dimensional Brownian motion). For a Bessel process one can make use of time inversion via the identity in distribution

$$(X_t : t > 0) = (tX_{1/t} : t > 0)$$

and potential theory for three dimensional Brownian motion which gives

$$P(X_s < \theta \text{ for some } s \geq 0 | X_0 = x) = \min\{\theta x^{-1}, 1\}.$$

Then

$$\begin{aligned} P(X_s < \theta s \text{ for some } s \leq \sigma) &= P(X_s < \theta \text{ for some } s \geq 1/\sigma) \\ &= E(\min\{\theta X_{1/\sigma}^{-1}, 1\}) \\ &= E(\min\{\theta \sigma^{1/2} X_1^{-1}, 1\}) \\ &= \int_0^\infty (2\pi)^{-3/2} r^2 \exp(-r^2/2) \min\{\theta\sigma^{1/2} r^{-1}, 1\} dr \\ &\leq C\theta\sigma^{1/2} \end{aligned}$$

where the penultimate equality follows from Brownian scaling and the final equality from a calculation using the density of the Gaussian variable X_1. To exploit this idea we divide the interval $[0, \sigma]$ into two parts, over the first of which the approximation $f(t,x) \approx 1/x$ is sufficiently good.

We first estimate $P(X_s < \theta s \text{ for some } s \leq \sigma/2)$. Using the elementary inequalities $(1 - z)/2z \leq (e^{2z} - 1)^{-1} \leq 1/2z$ for all $z > 0$ one obtains the bounds $x^{-1} - x(\sigma - t)^{-1} \leq f(t,x) \leq x^{-1} + 2m\sigma^{-1}$. Hence

$$X_t^{-1} - 2\sigma^{-1} X_t \leq f(t, X_t) \leq X_t^{-1} + 2m(\sigma - t)^{-1} \qquad \text{for } t \leq \sigma/2. \tag{53}$$

So the solution of the equation

$$dY_t = Y_t^{-1} dt - 2\sigma^{-1} Y_t dt + dW_t, \;\; Y_0 = 0$$

satisfies $Y_t \leq X_t$ for all $t \leq \tau$. To remove the unwanted $-2\sigma^{-1}Y_t dt$ in the drift of (Y_t) we use a change of measure. Define a new probability measure Q by defining the Radon-Nicodym derivative M by

$$\begin{aligned} M &= \left.\frac{dQ}{dP}\right|_{\mathcal{F}_{\sigma/2}} \\ &= \exp(2\sigma^{-1}\int_0^{\sigma/2} Y_s dW_s - 2\sigma^{-2}\int_0^{\sigma/2} Y_s^2 ds) \\ &= \exp(\sigma^{-1}Y_{\sigma/2}^2 + 2\sigma^{-2}\int_0^{\sigma/2} Y_s^2 ds - 3/2) \\ &\geq \exp(-3/2). \end{aligned}$$

The second equality here follows from Ito's formula. By Girsanov's theorem (see [29]) the process (Y_t) solves $dY = Y^{-1}dt + d\tilde{W}$ with respect to some Brownian motion $(\tilde{W})$ under Q, implying that (Y_t) is a three dimensional Bessel process under Q. Writing E_Q for the expectation under Q we have

$$\begin{aligned} P(X_s < \theta s \text{ for some } s \leq \sigma/2) &\leq P(Y_s < \theta s \text{ for some } s \leq \sigma/2) \\ &= E_Q(M^{-1}I(Y_s < \theta s \text{ for some } s \leq \sigma/2)) \\ &\leq e^{3/2}Q(Y_s < \theta s \text{ for some } s \leq \sigma/2) \\ &\leq C\theta\sigma^{1/2} \end{aligned} \tag{54}$$

using the argument given above.

It remains to estimate the probability $P(X_s < \theta s \text{ for some } \sigma/2 \leq s \leq \sigma)$. We shall further condition on the value of $X_{\sigma/2}$. If $X_{\sigma/2} \in dr$ the evolution of $(X_s : s \in [\sigma/2, \sigma])$ is that of a Brownian bridge starting at r, ending at m and conditioned to take non-negative values. We write Q_x for the law of a one-dimensional Brownian motion (W_t) started at x and we define $H_a = \inf\{t : W_t \leq a\}$. Then, supposing $r, q \geq \theta\sigma$, we have

$$\begin{aligned} & P(X_s \leq \theta s \text{ for some } s \in [\sigma/2, \sigma] | X_{\sigma/2} \in dr) \\ = \; & 1 - P(X_s > \theta s \text{ for all } s \in [\sigma/2, \sigma] | X_{\sigma/2} \in dr) \\ \leq \; & 1 - P(X_s > \theta\sigma \text{ for all } s \in [\sigma/2, \sigma] | X_{\sigma/2} \in dr) \\ = \; & 1 - Q_r(H_{\theta\sigma} > \sigma/2 | W_{\sigma/2} \in dm, H_0 > \sigma/2) \\ = \; & 1 - \frac{Q_r(H_{\theta\sigma} > \sigma/2, W_{\sigma/2} \in dm)}{Q_r(H_0 > \sigma/2, W_{\sigma/2} \in dm)} \end{aligned}$$

The reflection principle can be used to show that, for $a \leq r, m$,

$$Q_r(H_a > t, W_t \in dm) = (p_t(m - r) - p_t(m + r - 2a))\, dm \tag{55}$$

where $p_t(z) = (2\pi t)^{-1/2}\exp(-z^2/2t)$. Using this we rewrite the last expression as

$$\frac{\exp((m + r)^2/\sigma) - \exp((m + r - 2\theta\sigma)^2/\sigma)}{\exp((m + r)^2/\sigma) - \exp((m - r)^2/\sigma)}$$

$$
\begin{aligned}
&= (1-\exp(\frac{-4mr}{\sigma}))^{-1}4\theta(m+r-2\eta)\exp(\frac{(m+r-2\eta)^2-(m+r)^2}{\sigma}) \\
&\qquad\qquad \text{for some } \eta \in [0,\theta\sigma] \text{ by the mean value theorem} \\
&\leq (1-\exp(\frac{-4mr}{\sigma}))^{-1}4\theta(m+r-2\eta) \\
&\leq C\theta(1+\frac{\sigma}{4mr})(m+r) \qquad (\text{using } (1-e^{-z})^{-1} \leq C(1+z^{-1})) \\
&\leq C\theta(m+r+\sigma r^{-1}+\sigma m^{-1}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& P(X_s \leq \theta s \text{ for some } s \in [\tau,\sigma] | X_\tau \in dr) \\
\leq\ & C\theta(m+r+\sigma r^{-1}+\sigma m^{-1}) + I(r \leq \theta\sigma) + I(m \leq \theta\sigma). \qquad (56)
\end{aligned}
$$

We now undo the conditioning on $X_{\sigma/2} \in dr$. Using the upper bound in (53) and Ito's formula one obtains $dX_t^2 \leq (3+4m\sigma^{-1}X_t)dt + 2X_t dW_t$. Taking expectations one has

$$
\begin{aligned}
E(X_t^2) &\leq 3t + 4m\sigma^{-1}\int_0^t E(X_s)ds \\
&\leq (3+m^2\sigma^{-1})t + 4\sigma^{-1}\int_0^t E(X_s^2)ds.
\end{aligned}
$$

Applying Gronwall's inequality shows that $E(X_{\sigma/2}) \leq (E(X_{\sigma/2}^2))^{1/2} \leq C(\sigma^{1/2}+m)$. By Markov's inequality $P(X_\tau \leq \theta\sigma) \leq \theta\sigma E(X_\tau^{-1})$. Using the comparison with a Bessel process as before we have $E(X_\tau^{-1}) \leq e^{3/2}E_Q(Y_{\sigma/2}^{-1}) \leq C\sigma^{-1/2}$. Using these bounds in (56) and combining with (54) leads to the estimate in part d) of the lemma.

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